Midterm Classwork: Examples

Exercises

Exercise 0.1. I have 3 umbrellas, and each of them may be at home or at the office. Each time I go from home to office, or from the office to home, I take an umbrella with me iff it rains. However, if there are no umbrellas available and it rains, I get wet. The probability that it rains during a given trip office/home or home/office is p. Find the probability that I get wet on a given trip.

Solution

The time t here represents the number of places (office/home) visited. We let X_t be the number of umbrellas that are in the place where I am. This is a Markov chain with state space $S = \{0, 1, 2, 3\}$. If $x = 0$, then $p_{0,3} = 1$, since once I change place, all the umbrellas will be at the new place. $p_{1,3} = p$ (since I will bring with me the only available umbrella iff it rains), $p_{1,2} = 1 - p$; and similarly $p_{2,2} = p = 1 - p_{2,1}$; finally $p_{3,1} = p = 1 - p_{3,0}$. In other words, for $x \geq 1$, $p_{x,y} = p$ if $y = 4 - x$ (since I will bring the umbrella with me), and $p_{x,y} = 1 - p$ if $y = 3 - x$.
The transition probabilities matrix is then

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1-p & p \\ 0 & 1-p & p & 0 \\ 1-p & p & 0 & 0 \end{pmatrix}$$

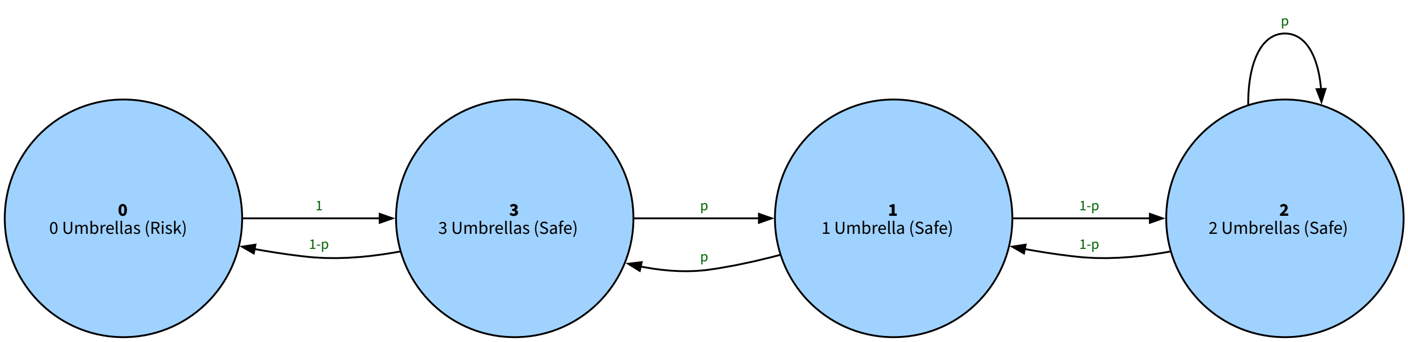

Figure 1

This is an irreducible Markov chain, with unique invariant probability. It is easy to see that $m_3 = m_2 = m_1$. So using $mP = m$ we get $m = (\frac{1-p}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}, \frac{1}{4-p})$. The probability that I get wet is then $m_0 p = \frac{p(1-p)}{4-p}$. This probability is maximized for $p = 2(2 - \sqrt{3}) \approx 0.54$, aka Moscow weather. Notice also that, if I get N umbrellas, the probability of getting wet just becomes $p(1-p)/(N+1-p) \leq 1/(4N)$.

Exercise 0.2. Damocles was a courtier at the court of the tyrant Dionysius of Syracuse. Dionysius gives him a task to complete every now and then and, depending on the outcome, the consideration the tyrant has for Damocles changes. However over time there are only two possible outcomes:

- Damocles becomes a favorite at the court and given a high title. This earns him a drachmas.
- Damocles is banished and deprived of anything he has, worth b drachmas.

We can model the reputation of Damocles as a finite state homogeneous Markov chain, which sooner or later will hit either the set A (achieving the high title) or B (banishment). E.g. we can think the reputation is an integer numerical value x, ranging from 0 (banishment) to 100 (high title).

a. Assuming we know the transition probabilities, write a linear problem to determine the expected amount of drachmas earned by Damocles when starting his service with a given reputation $x \in S$.

At some point, Dionysius famously hangs a sword on the head of Damocles, held by a single horsehair. Before he completes each task, the sword has a probability $p > 0$ of falling on the head of Damocles (killing him).

b. Write a linear problem to determine the expected amount of drachmas earned by Damocles when starting his service with a given reputation $x \in S$, with the sword hanging on his head.

> **Solution**
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> Let τ be the first time when Damocles reputation is decided (meaning he gets a high title or is banished). Conditioning on the first step, the function $h(x) = \mathbb{E}[a\mathbf{1}_{X_\tau \in A} - b\mathbf{1}_{X_\tau \in B}]$ solves
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> $$\begin{aligned} &(I-P)h = 0 \qquad \text{on } (A \cup B)^c \\ &h = a \qquad \text{on } A \\ &h = -b \qquad \text{on } B \end{aligned}$$
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> If we add the Damocles' sword, for instance we can think that we add an extra point ξ to the state space, say $S' = S \cup \{\xi\}$. If $(p_{x,y})$ are the original transition probabilities, the new Markov chain has transition probabilities given by
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> $$\begin{aligned} &q_{x,\xi} = p, \qquad x \in S \\ &q_{x,y} = (1-p)p_{x,y} \qquad x, y \in S \\ &q_{\xi,\xi} = 1 \end{aligned}$$
>
> We have the same formula for h, just τ is now the first time where the chain hits $A \cup B \cup \{\xi\}$. $h(x)$ now satisfies, for $x \in S \setminus (A \cup B)$, $h(x) = \sum_{y \in S} q_{x,y} h(y) + 0 q_{x,\xi}$. Namely, again as an equation on S (ξ was just auxiliary):
>
> $$\begin{aligned} &(I-(1-p)P)h = 0 \qquad \text{on } (A \cup B)^c \\ &h = a \qquad \text{on } A \\ &h = -b \qquad \text{on } B \end{aligned}$$

Exercise 0.3. For a homogeneous Markov Chain, explain why each point in a non-closed communicating class C is transient.

> **Solution**
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> Since C is not closed, there is $y \in C$, $z \notin C$ such that $p_{y,z} > 0$. But $x \leftrightarrow y$ and $z \nrightarrow x$. Let t be the smallest time such that $p^{(t)}_{x,y} > 0$. Then $q := p^{(t+1)}_{x,z} \geq p^{(t)}_{x,y} p_{y,z} > 0$. Then each time the chain touches x, it has probability at least $q > 0$ to never return. Thus it will only return finitely many times.

Exercise 0.4. An irreducible Markov chain on $\mathbb{N}$ has the following feature. For each $x \in S$, $p_{x,0} \geq p$ for some $p > 0$.

a. Prove that $\mathbb{E}_x[\tau_x^+] < \infty$.
b. Deduce (using known results) that there exists a unique invariant probability m.
c. Estimate

$$\sup_x |\mathbb{P}_y(X_t = x) - m_x| \leq ?$$

Solution

a. Fix x. By irreducibility, for some t, we have $p_{0,x}^{(t)} > 0$. Thus at every step we have probability at least p of jumping to 0 in one step, and then going back to x in t steps:

$$\begin{aligned} &\mathbb{P}_0(\tau_x^+ \leq t) \geq p_{0,x}^{(t)} \\ &\mathbb{P}_y(\tau_x^+ \leq t+1) \geq p\, p_{0,x}^{(t)} := q_x > 0 \\ &\mathbb{P}_x(\tau_x^+ \geq k(t+1)) \leq (1-q_x)^k \end{aligned}$$

and therefore its expectation is finite.

b. Every irreducible, positive-recurrent Markov chain admits a unique invariant probability m.

c. Let Y_t be an independent chain, started with initial distribution m. Let τ be the first time where $X_t = Y_t$. We have that

$$|\mathbb{P}_y(X_t = x) - m_x| \leq \mathbb{P}(\tau > t) \leq (1-p^2)^t$$

With more care, we can construct a coupling (X_t, Y_t) where both chains jump to 0 simultaneously with probability p at each step. To achieve this, we flip a coin with probability p of landing on *heads*. If it lands on *heads*, both X_t and Y_t move to 0. If it lands on *tails*, they proceed independently with modified transition probabilities (specifically, from state x, the chain jumps to 0 with probability $(p_{x,0}-p)/(1-p)$). This construction yields the improved bound $(1-p)^t$.