Midterm Classwork (Retake)

- You have 1 hour and 50 minutes.
- You can check your own notes and books, and the lectures notes on the shared tablet.
- You cannot use your own electronics.

Exercises

Exercise 0.1. Let $\mathbf{X} = (X_t)_{t\geq 0}$ be a Markov chain on the two-state space $S = \{1, 2\}$ with transition matrix:

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

where $\alpha, \beta \in (0, 1)$.

a. Find the unique invariant distribution m of this chain.

b. Fix $T > 1$. Let $Y_t = X_{T-t}$ be the time-reversed chain starting from the stationary distribution. Determine the transition matrix of $\mathbf{Y}$.

c. If μ_t is the law of X_t when starting with initial measure $\mu_0 \neq m$. Determine

$$\lim_{t\to\infty} \frac{1}{t} \log \|\mu_t - m\|_{TV}$$

> Solution
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> a. $m_1 = \frac{\beta}{\alpha+\beta}$, $m_2 = \frac{\alpha}{\alpha+\beta}$.
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> b. The chain is reversible, so it has the same transition probabilities.
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> c. P has eigenvalues 1 and $1-\alpha-\beta$. The limit is therefore $-\log(1-\alpha-\beta)$ (which is $-\infty$ if $\alpha+\beta = 1$, since X_t becomes i.i.d. in this case, for $t \geq 1$).

Exercise 0.2. A bug moves on a line segment with sites $S = \{a, b, c\}$. Site a is a "reflecting wall": if the bug is at a, it moves to b with probability 1 at the next step. Site c is a "sticky trap" (absorbing state): if the bug is at c, it stays at c forever. At site b, the bug moves to c with probability $p \in (0, 1)$ and moves back to a with probability $1 - p$. Find the expected number of steps needed for the bug to reach the trap when starting at a.

> Solution
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> Let τ_c be the hitting time of c, which is finite a.s., and let $u(x) := \mathbb{E}[\tau_c]$. u satisfies $u(c) = 0$ and $(I - P)u = 1$. Namely $u(b) = 1 + (1-p)u(a)$, $u(a) = u(b) + 1$. Thus $u(a) = 2/p$.

Exercise 0.3. Let $S = \mathbb{Z} \times \mathbb{Z}_N$, where $\mathbb{Z}_N := \{0, 1, \dots, N\}$ and $i, j \in \mathbb{Z}_N$ are neighbors if $i = j \pm 1 \pmod N$.

a. Determine whether the simple random walk on S is recurrent or transient.
b. For $N = 2$, determine the (positive, finite at each point) invariant measures for such a random walk.

Solution

a. Let $X_t = (Y_t, Z_t)$ be the random walk. $\mathbf{X}$ is irreducible, therefore each point have the same transient/recurrent property. $\mathbf{Y} = (Y_t)$ is a random walk on $\mathbb{Z}$, with transition probabilities $q_{y,y} = 1/2$, $q_{y,y\pm1} = 1/4$. Thus $\mathbf{Y}$ is recurrent and it visits each point, e.g. 0, infinitely many times with probability 1. Each time Y_t visits 0, X_t has strictly positive probability of visiting $(0, 0)$, therefore $\{t : X_t = (0, 0)\}$ is infinite with probability 1. Namely $\mathbf{X}$ is recurrent.
b. Uniform measures $m_x = c$ are the only invariant ones.