

# Martingales

We briefly recall some basic properties and inequalities concerning martingales.

## Definitions

**Definition 0.1** (Filtered space). Let  $\Theta$  be a partially ordered set and  $(\Omega, \mathcal{F})$  a **measurable space**.

A **filtration** on  $\Omega$  indexed by  $\Theta$ , is an increasing collection  $(\mathcal{F}_{t \in \Theta})_{t \in \Theta}$  of sub- $\sigma$ -algebras of  $\mathcal{F}$ . Namely for any  $s, t \in \Theta$  with  $s \leq t$ ,  $\mathcal{F}_s \subset \mathcal{F}_t$ . A filtration is **separable**<sup>1</sup> if there is a countable set  $\Theta' \subset \Theta$  such that  $\mathcal{F}_t = \bigcap_{s \in \Theta', s > t} \mathcal{F}_s$  for all  $t \in \Theta$ .

The triple  $(\Omega, \mathcal{F}, \mathcal{F}_{t \in \Theta})$  is then called a **filtered space**.

If  $\mathbb{P}$  is a probability on  $(\Omega, \mathcal{F})$ ,  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \Theta}, \mathbb{P})$  is called a **filtered probability space**. It is a **complete filtered probability space** if  $(\Omega, \mathcal{F}_t, \mathbb{P})$  is **complete** for every  $t \in \Theta$ .

**Definition 0.2.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \Theta})$  be a filtered space,  $S$  a measurable space and  $\mathbf{X} = (X_t)_{t \in \Theta}$  a collection of measurable maps  $X_t: \Omega \rightarrow S$ .  $\mathbf{X}$  is **adapted** to  $\mathcal{F}_t$  if  $X_t$  is  $\mathcal{F}_t$ -measurable for  $t \in \Theta$ . The **natural filtration**  $\mathcal{F}_t^{\mathbf{X}}$  of  $\mathbf{X}$  is the weakest filtration  $\mathbf{X}$  is adapted to.

**Definition 0.3** (Martingale). Let  $(\Omega, \mathcal{F}, \mathcal{F}_{t \in \Theta})$  be a filtered space, and for  $t \in \Theta$  let  $M_t$  be a real-valued random variable  $M_t: \Omega \rightarrow \mathbb{R}$  with  $M_t \in L^1(\mathbb{P})$ , namely  $\mathbb{E}[|M_t|] < \infty$ , for  $t \in \Theta$ .

For  $\mathbb{P}$  a probability on  $(\Omega, \mathcal{F})$ ,  $\mathbf{M} = (M_t)_{t \in \Theta}$  is

- a  **$\mathbb{P}$ -submartingale** if  $\mathbb{E}[M_t | \mathcal{F}_s] \geq M_s$ , for  $s, t \in \Theta$  with  $s \leq t$ .
- a  **$\mathbb{P}$ -supermartingale** if  $\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s$ , for  $s, t \in \Theta$  with  $s \leq t$ .
- a  **$\mathbb{P}$ -martingale** if it is both a supermartingale and a submartingale, namely if  $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$ , for  $s, t \in \Theta$ .

If  $\mathbb{P}$  is understood from the context, then it is usually omitted in the notation, for instance a  $\mathbb{P}$ -martingale is just called martingale in this case.

**Example 0.1.** Let  $(X_i)_{i \geq 1}$  be a sequence of independent real-valued integrable random variables, and  $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$  be the smallest  $\sigma$ -algebra such that  $(X_1, \dots, X_n)$  is measurable.

<sup>1</sup>When  $\Theta$  is a totally ordered, separable space, in particular when  $\Theta \subset \mathbb{R}$ , a separable filtration is also called **right-continuous**. Indeed in this case one usually introduces the right-augmented filtration

$$\mathcal{F}_t^+ := \bigcap_{s > t} \mathcal{F}_s$$

and separability is equivalent to  $\mathcal{F}_t = \mathcal{F}_t^+$ .

Then  $M_n := \sum_{i=1}^n X_i$  is a submartingale iff  $\mathbb{E}[X_i] \geq 0$  for all  $i \geq 1$ , a supermartingale iff  $\mathbb{E}[X_i] \leq 0$  for all  $i$ . And thus a martingale iff  $\mathbb{E}[X_i] = 0$  for all  $i$ . Indeed, for  $k < n$ , using independence

$$\mathbb{E}[M_n | \mathcal{F}_k] = \mathbb{E}[M_k + (X_{k+1} + \dots + X_n) | (X_1, \dots, X_k)] = M_k + \sum_{i=k+1}^n \mathbb{E}[X_i]$$

So for instance if the expectations are non-negative,  $M_n$  is a submartingale. On the other hand, if  $M_n$  is a submartingale, then one needs  $\mathbb{E}[X_n] \geq 0$  for all  $n \geq 1$ , taking  $k = n - 1$  above.

*Remark 0.1.*  $M_t \in L^1(\mathbb{P})$  is a martingale w.r.t. a given filtration iff it is a martingale w.r.t. its natural filtration, namely iff  $\mathbb{E}[M_t | (M_r)_{r \leq s}] = M_s$  for  $s \leq t$ .

*Remark 0.2.* If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is convex (concave) and  $M_t$  is a submartingale (supermartingale), then  $f(M_t)$  is a submartingale (supermartingale), provided  $f(M_t) \in L^1(\mathbb{P})$ . This is a consequence of [Jensen inequality](#). In particular linear combinations of martingales are martingales.

*Remark 0.3.* If  $\Theta = \mathbb{N}$  (or any other countable set), the filtration is automatically separable, and a martingale (sub/supermartingale)  $(M_n)$  is called a **discrete** martingale (sub/supermartingale) in this case.

*Remark 0.4.* If  $\Theta = [0, \infty)$  (or some other real interval), the filtration is separable iff

$$\mathcal{F}_s = \mathcal{F}_s^+ := \bigcap_{t > s} \mathcal{F}_t$$

since we can always take the intersection on the rational numbers. In this case, the filtration is usually called right-continuous. A martingale (sub/supermartingale)  $(M_t)_{t \geq 0}$  is called a **continuous-time** martingale (sub/supermartingale) in this case.

If  $(M_n)$  is a discrete martingale (sub/supermartingale) on a discrete filtered space  $(\Omega, \mathcal{F})$ ,  $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mathbb{P})$ , we can define for  $t \in [n, n + 1)$ :

$$\mathcal{F}_t := \mathcal{F}_n \quad M_t := M_n$$

to get a continuous-time martingale (sub/supermartingale). So the theory for continuous-time martingales covers indeed the theory for discrete martingales.

## Continuous-time martingales and Stopping times

### Doob convergence Theorem

For a real number  $a \in \mathbb{R}$ , we use the notation  $a^+ := \max(x, 0)$ ,  $a^- := \max(-a, 0)$ , so that  $a = a^+ - a^-$  and  $|a| = a^+ + a^-$ . For  $\mathbf{X} := (X_t)_{t \in \mathbb{R}}$  we also write

$$X_{t^+} := \lim_{s \downarrow t} X_s, \quad X_{t^-} := \lim_{s \uparrow t} X_s$$

and we define the random sets<sup>2</sup>

$$I^+ \equiv I_+(\mathbf{X}) := \{t \in \mathbb{R} : X_t = X_{t^+}\}, \quad I_- \equiv I^-(\mathbf{X}) := \{t \in \mathbb{R} : X_t = X_{t^-}\}$$

**Theorem 0.1.** Let  $\mathbf{X} := (X_t)_{t \in \mathbb{R}}$  be a continuous time supermartingale such that

$$\overline{\lim}_t \mathbb{E}[X_t^-] < \infty \tag{1}$$

<sup>2</sup>to be precise, in the notation  $X_t = X_{t^+}$  we mean that the limit must exist and equal  $X_t$ , or equivalently  $\overline{\lim}_{s \downarrow t} |X_s - X_t| = 0$ .

Then  $X_t$  admits left and right limits, with probability 1:

$$\mathbb{P}(X_{t+}, X_{t-} \text{ exist for all } t) = 1$$

Moreover there exists a random variable  $X \in L^1(\Omega)$  such that

$$\lim_{t \rightarrow \infty, t \in I_+ \cup I_-} X_t = X \quad \text{a.s.}$$

In particular, under Equation 1, a right-continuous (or left-continuous) supermartingale admits an a.s. limit.